Axler 7B

Definition of L^p: Let $Z(\mu) = \{ f = 0 \text{ a.e.} \}$

Let $\tilde{f} = \{ f + z : z \in Z \}$ (the equivalence class of f)

Note: 1) $\tilde{f} = \tilde{g}$ iff $f = g$ a.e.

2) How to add equivalence classes?

$\tilde{f} + \tilde{g} := \widetilde{(f+g)}$ $\quad \alpha \tilde{f} := \widetilde{(\alpha f)}$

well defined.

3) $\| \tilde{f} \|_p = \| f \|_p$ (again well defined.

L^p is a Banach space (Riesz-Fisher theorem)

Let (X, S, μ) $1 \leq p \leq \infty$, $\{f_n\}$ Cauchy in $\mathcal{L}^p$. Then $\exists f \in \mathcal{L}^p$ st $f_n \to f$.

Pf: $p = \infty$ easy. Let $1 \leq p < \infty$. Choose a subseq st

$$\| f_{n_{k+1}} - f_{n_k} \| < \frac{1}{2^k} \Rightarrow \sum \| f_{n_{k+1}} - f_{n_k} \| < \infty$$

Let $g_m = \sum_{k=1}^{m} | f_{n_{k+1}}(x) - f_{n_k}(x) |$ $\quad g(x) = \sum_{k=1}^{\infty} | f_{n_{k+1}}(x) - f_{n_k}(x) |$

Monotonicity $\Rightarrow$ $g_m(x) \uparrow g(x)$ $\forall x$

Minkowski $\Rightarrow$ $\| g_m \|_p \leq \sum_{k=1}^{\infty} \| f_{n_{k+1}} - f_{n_k} \|$

MCT $\Rightarrow$ $\int g^p = \lim_{m \to \infty} \int g_m^p \leq \sum_{k=1}^{\infty} \| f_{n_{k+1}} - f_{n_k} \|$ so $g(x) < \infty$ a.e.

So $f(x) = \sum_{k=1}^{\infty} f_{n_{k+1}} - f_{n_k}(x) + f_{n_1}(x)$ is summable for ae x. Using completeness of F. Moreover $|f(x)| \leq g(x)$ a.e. x

$$\| f_{n_k} - f \|_p^p = \int | f_{n_k} - f |^p = \int \lim_{j \to \infty} | f_{n_k} - f_{n_j} |^p$$

$$\leq \underline{\lim}_{j \to \infty} \int | f_{n_k} - f_{n_j} |^p < \epsilon^p$$

$\Rightarrow$ $f_{n_k} \to f$ in $\mathcal{L}^p$ norm $\Rightarrow$ $f_n \to f$ in $\mathcal{L}^p$ (since is Cauchy)

Rem: Proof also shows if f_n is Cauchy in $\mathcal{L}^p$, then f_n has a pointwise a.e. convergent subsequence.

Duality: Suppose $1 < p \leq \infty$ $h \in L^q$ $\varphi_h : L^p \to F$

$\varphi_h(f) = \int f h$ $\quad \| \varphi_h(f) \| \leq \| f \|_p \| h \|_q$ Hölder

$\Rightarrow$ φ_h is bounded. In fact, $\| \varphi_h \|_{op} = \| h \|_q$ $\quad q = \frac{p}{p-1}$

$\Rightarrow q - 1 = \frac{1}{p-1}$

Pf: Choose $f = \dfrac{h^{q-1} \operatorname{sgn}(h)}{\| h \|_q^{q-1}}$

$\int f h = \| h \|_q$ $\quad \| f \|_p^p = \int \dfrac{|h|^{(q-1)p}}{\| h \|_q^{(q-1)p}} = \dfrac{\int |h|^q}{\| h \|_q^q} = 1$

If $h \in L^\infty$ then may not work.